TENSOR PRODUCTS OVER BANACH ALGEBRAS

BY

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0. Introduction. In [4], [5], [6] the structures $A_1 \otimes_\gamma A_2$, where A_1 and A_2 are Banach algebras, are discussed. Actually, a proper parallel to the algebraic situation is: three commutative Banach algebras A, B, C, where A and B are C-bimodules (in the sense described below), and some Banach-algebraic version of $A \otimes_C B$. In the first part of the following, we shall give a general discussion of $A \otimes_C B$, a natural Banach-algebraic version of the (algebraic) tensor product of A and B over C. Thereafter, we shall discuss a special case in which A, B, C are group algebras of locally compact abelian groups. Finally we shall handle the even more special problem in which A and B are group algebras of locally compact abelian groups and C is a group algebra of a compact abelian group. In the last two parts, particularly the last, a connection will be established between the theory of tensor products and the theory of group extensions. In this connection, the author thanks L. Auslander for indicating the point in question. The author is also indebted to Dr. B. Natzitz for pointing out errors and the need for clarifying several items of an earlier draft of this paper.

1. $A \otimes_C B$. In analogy to the treatment in [5], [6] we consider three commutative Banach algebras A, B, C where it is assumed that A and B are C-bimodules. The fundamental topological assumption is: For $(a, b, c) \in A_1 \times B \times C$, $\| ac \| \leq \| a \| \| c \|$ and $\| bc \| \leq \| b \| \| c \|$.

We construct the linear function space and commutative topological algebra

$$F_C(A, B) = \left\{ f \mid f \in C^{A \times B}, \ \gamma_1(f) \equiv \sum_{(a,b)} \| f(a,b) \| \| a \| \| b \| < \infty \right\}$$

where multiplication $*$ of two elements f_1, f_2 is defined by

$$(f_1 * f_2)(a, b) = \sum_{a_1 a_2 = a; b_1 b_2 = b} f_1(a_1, b_1) f_2(a_2, b_2), \quad \text{if } \| a \| \| b \| > 0,$$

$$= 0, \qquad \qquad \text{otherwise.}$$

The relations

$$\sum_{(a,b)} \left(\sum_{a_1 a_2 = a; b_1 b_2 = b} \| f_1(a_1, b_1) f_2(a_2, b_2) \| \right) \| a \| \| b \|$$

$$\leq \sum_{a_1, a_2, b_1, b_2} \| f_1(a_1, b_1) \| \| f_2(a_2, b_2) \| \| a_1 \| \| a_2 \| \| b_1 \| \| b_2 \|$$

$$= \gamma_1(f_1) \gamma_1(f_2) < \infty$$

show that the series defining $(f_1 * f_2)(a, b)$ converges whenever $\|a\| \|b\| > 0$. Thus $f_1 * f_2$ is well defined and is in $F_C(A, B)$. Commutativity and associativity of $*$ are easily proved.

In F_C we consider the closed ideal I generated by the functions of the following types:

(1)
$$f(a_1 + a_2, b_1) = -f(a_1, b_1) = -f(a_2, b_1),$$
$$f(a, b) = 0, \text{ otherwise};$$

(2)
$$f(a_1, b_1 + b_2) = -f(a_1, b_1) = -f(a_1, b_2),$$
$$f(a, b) = 0, \text{ otherwise};$$

(3)
$$f(a_1 c_1, b_1) = -f(a_1, c_1 b_1),$$
$$f(a, b) = 0, \text{ otherwise};$$

(4)
$$f(a_1 c_1, b_1) c_1 = -f(a_1, b_1),$$
$$f(a, b) = 0, \text{ otherwise}.$$

(If C has no identity, then for $\lambda \in C$ we include in I the functions

(3')
$$f(a_1 \lambda, b_1) = -f(a_1, \lambda b_1),$$
$$f(a, b) = 0, \text{ otherwise};$$

(4')
$$f(a_1 \lambda, b_1) \lambda = -f(a_1, b_1),$$
$$f(a, b) = 0, \text{ otherwise}.)$$

We then define $A \otimes_C B$ to be $F_C(A, B)/I$. After the next remarks we shall show that $D = A \otimes_C B$ is a commutative Banach algebra and then we shall investigate the maximal ideal space $\mathfrak{M}_D$.

It is of interest first to illustrate the above concepts.

Example 1. Let $S \equiv \{z \,|\, |z| \leq 1\}$ and let $A = B = C(S)$. Let $C = A_1 \equiv \{f \,|\, f$ analytic in S^0, f continuous on $S\}$. If A and B are regarded as C-bimodules with elements of C acting by pointwise multiplication, then the conditions of our general discussion prevail and $A \otimes_C B$ is definable.

Example 2. Let G, H, K be locally compact abelian groups, and let $\theta_G : K \to G$, $\theta_H : K \to H$ be homomorphisms such that $\theta_G(K)$ and $\theta_H(K)$ are closed subgroups of G and H respectively. Let $A = L_1(G)$, $B = L_1(H)$ and $C = L_1(K)$ and for $a \in A$, $c \in C$ define ac by

$$ac(\xi) = \int_K a(\xi - \theta_G(\zeta)) c(\zeta) \, d\zeta .$$

Again (see §3) the definitions permit the construction of $A \otimes_C B$. (Note the case: G, H subgroups of a supergroup, $G \supset K$, $H \supset K$, θ_G, θ_H the inclusion maps.)

Example 3. Let $H^\infty = \{f \mid f$ analytic in S^0, $\sup_{S^0} |f| < \infty\}$. Then if (a) $A = B = C(S)$, $C = H^\infty$ or if (b) $A = B = H^\infty$ and $C = A_1$, $A \otimes_C B$ is definable in a standard way.

Example 4.   Let $C^{(k)}([0, 1])$ be the set of functions continuous on $[0, 1]$ and having k continuous derivatives on $[0, 1]$. For $f \in C^{(k)}([0, 1])$ let

$$\|f\| = \sum_{j=1}^{k} \sup \{|f^{(j)}(x)| \mid 0 \leq x \leq 1\}.$$

Then with the obvious definitions of operations, if $0 \leq k_1$, $k_2 \leq k_3$ and if $A = C^{(k_1)}([0, 1])$, $B = C^{(k_2)}([0, 1])$, $C = C^{(k_3)}([0, 1])$, we see that $A \otimes_C B$ is definable.

Example 5. A and B are two commutative subalgebras of a commutative Banach algebra and $C = A \cap B$.

In defining $A \otimes_C B = D$, we did not prove it is a Banach space. However, the only question of substance is resolved by

LEMMA 1.   *If $f \in F_C(A, B)$ and if $\gamma_1(f) = 0$, then $f \in I$.*

Proof.   For each pair (a, b) either $\|a\| = 0$ or $\|b\| = 0$ or $f(a, b) = 0$. We may confine ourselves to the case $\|a\| = 0$. Let $f(a, b) = c$ and choose $a_1 \neq 0$. Then using the customary symbolism we find:

$$c((a_1 - a_1) \otimes b) - c(a_1 \otimes b) - c((-a_1) \otimes b) \in I, \text{ via } (1);$$

$$- c(a_1 \otimes b) - c((-a_1) \otimes b) \in I, \text{ via } (4');$$

and thus $c(0 \otimes b) \in I$.

Thus, from Lemma 1 we can conclude that if $\gamma(z)$ denotes the quotient seminorm of $z \in D$, then $\gamma(z) = 0$ if and only if $z = 0$. It is now clear that D is a commutative Banach algebra.

2. **Maximal ideals.**   Let $\mathfrak{M}_C$ be the set of multiplicative homomorphisms of C into C (or, equivalently, the set of regular maximal ideals of C). Let $\mathfrak{M}_C^0$ be $\mathfrak{M}_C$ with the null map adjoined. We use $\mathfrak{M}_A$ and $\mathfrak{M}_B$ in a similar sense.

THEOREM 1.   *There is a homeomorphism $\tau : \mathfrak{M}_D \to \mathfrak{M}_A \times \mathfrak{M}_B$ and the image of τ is a locally compact subset of $\mathfrak{M}_A \times \mathfrak{M}_B$. There are continuous mappings $\mu : \mathfrak{M}_A \to \mathfrak{M}_C^0$, $v : \mathfrak{M}_B \to \mathfrak{M}_C^0$ such that if $\rho = \mu \times v$ then $\tau(\mathfrak{M}_D) = \rho^{-1}(\Delta)$ where $\Delta = $ diagonal of $\mathfrak{M}_C \times \mathfrak{M}_C$. These mappings are furthermore related as follows: If $z \in D$ and if z has the representative*

$$\sum_{n=1}^{\infty} c_n(a_n \otimes b_n) \qquad [5], [6],$$

then for any $M_D \in \mathfrak{M}_D$

$$\hat{z}(M_D) = \sum_{n=1}^{\infty} \hat{c}_n(M_C)\hat{a}_n(M_A)\hat{b}_n(M_B),$$

where $\tau(M_D) = (M_A, M_B)$ *and* $\mu(M_A) = \nu(M_B) = M_C$.

The proof of the theorem will follow a sequence of lemmas.

LEMMA 1. *There is a function* $k(c, M_A)$, *independent of a such that for* $a \in A$, $[(ca)]^\wedge(M_A) = k(c, M_A)\hat{a}(M_A)$.

Proof. Let $a \notin M_A$ and let

$$k(c, a, M_A) = [(ca)]^\wedge(M_A)/\hat{a}(M_A).$$

We show that so long as $a \notin M_A$, $k(c, a, M_A)$ is independent of a. Indeed, if $a_1 \notin M_A$,

$$k(c, a, M_A) - k(c, a_1, M_A) = \frac{[(ca)]^\wedge(M_A)\hat{a}_1(M_A) - [(ca_1)]^\wedge(M_A)\hat{a}(M_A)}{\hat{a}(M_A)\hat{a}_1(M_A)}.$$

The numerator of the last fraction is $[(caa_1)]^\wedge(M_A) - [(ca_1a)]^\wedge(M_A)$ which is 0 since A is commutative. Dropping a from $k(c, a, M_A)$, we let the result denote the required function: $k(c, M_A)$. If $\hat{a}(M_A) \neq 0$, then clearly $[ca]^\wedge(M_A) = k(c, M_A)\hat{a}(M_A)$.

We observe that every $M_A \in \mathfrak{M}_A$ is a C-ideal and similarly every $M_B \in \mathfrak{M}_B$ is a C-ideal. Indeed, if $a \in M_A$, $c \in C$, let $a_n \notin M_A$, $a_n \to a$. Then $ca_n \to ca$, $[ca_n]^\wedge(M_A) = k(c, M_A)\hat{a}_n(M_A) \to 0$, i.e., $[ca]^\wedge(M_A) = 0$.

Hence, in particular, for all a: $[ca]^\wedge(M_A) = k(c, M_A)\hat{a}(M_A)$.

COROLLARY. *If for some* $a_0 \notin M_A$, $ca_0 \in M_A$, *then* $cA \subset M_A$.

Proof. $k(c, M_A) = 0$, whence for any $a \in A$, $[ca]^\wedge(M_A) = 0$.

LEMMA 2. *For* M_A *fixed,* $k(c, M_A)$ *is a continuous homomorphism of* C *into* C. (*Either there is an* $M_C \in \mathfrak{M}_C$ *such that* $\hat{c}(M_C) = k(c, M_A)$ *or* $k(c, M_A) \equiv 0$.)

Proof. The equations $k(c_1 + c_2, M_A) = k(c_1, M_A) + k(c_2, M_A)$ and $k(\lambda c_1, M_A) = \lambda k(c_1, M_A), \lambda \in C$, are clearly true. On the other hand $[(c_1 c_2)a]^\wedge(M_A) = k(c_1, M_A)(c_2 a)^\wedge(M_A) = k(c_1, M_A)k(c_2, M_A)\hat{a}(M_A)$, whence $k(c_1 c_2, M_A) = k(c_1, M_A)k(c_2, M_A)$. Finally, if $\lim_{n \to \infty} c_n = 0$, then for $a \notin M_A$

$$\left| k(c_n, M_A) \right| \leqq \frac{\|c_n\| \, \|a\|}{\hat{a}(M_A)} \to 0$$

as $n \to \infty$. Thus $k(c, M_A)$ is continuous in c.

Note that if $C = C$ then $k(c, M_A) = c$.

Defining $\mu: \mathfrak{M}_A \to \mathfrak{M}_C^0$ by $\mu(M_A) = M_C$ if $k(c, M_A) = \hat{c}(M_C)$ and $\mu(M_A) =$ null map if $k(c, \mathfrak{M}_A) \equiv 0$, and similarly defining $\nu: \mathfrak{M}_B \to \mathfrak{M}_C^0$, we shall define τ so that τ, μ, ν behave as stated in the theorem. (In Example 5, if C is contained in

the radicals of both algebras A and B, then $\mu(M_A) \equiv$ null map, $\nu(M_B) \equiv$ null map for all M_A, M_B. In other words $\rho(\mathfrak{M}_A \times \mathfrak{M}_B)$ may be completely outside $\mathfrak{M}_C \times \mathfrak{M}_C$.) First we prove

LEMMA 3. *μ and ν are continuous.*

Proof. Let $\mu(M_A) = X$, where X is either some M_C or X is the null map. If $N(X) = \{Y \in \mathfrak{M}_C^0 \mid |\hat{c}_i(Y) - \hat{c}_i(X)| < \varepsilon, \ i = 1, 2, \cdots, n\}$, let $\hat{a}(M_A) = 1$ and let $N(M_A) = \{M_A' \mid |[(ac_i)]^\wedge(M_A') - [(ac_i)]^\wedge(M_A)| < \varepsilon_1, \ |\hat{a}(M_A') - \hat{a}(M_A)| < \varepsilon_1, \ i = 1, 2, \cdots, n\}$. Then for $0 < \varepsilon_1 < \varepsilon/(1 + \max_i \|c_i\|)$, $\mu(N(M_A)) \subset N(X)$. Hence μ (and similarly ν) is continuous.

To define τ we proceed as follows. Let $M_D \in \mathfrak{M}_D$ and let $u \in D$ be such that $\hat{u}(M_D) = 1$. Regarding D as a bimodule with respect to A, B and C we define:

$$\phi_A(a) = [(au)]^\wedge(M_D),$$

$$\phi_B(b) = [(bu)]^\wedge(M_D),$$

$$\phi_C(c) = [(cu)]^\wedge(M_D) \qquad [5], [6].$$

These are clearly continuous homomorphisms into C. We shall show that ϕ_A and ϕ_B are surjections by first noting

LEMMA 4. *A regular ideal in D is simultaneously an A-ideal, B-ideal and C-ideal.*

Proof. See proof of Lemma 1 [5, §1].
If $\phi_A^{-1}(0) = A$, and if u is represented by

$$\sum_{n=1}^{\infty} c_n'(a_n' \otimes b_n'),$$

then, for any a_m',

$$\sum_{n=1}^{\infty} c_n'(a_n' a_m' \otimes b_n')$$

represents an element in M_D, whence $u^2 \in M_D$ and a contradiction results. Thus let $\phi_A^{-1}(0) = M_A$, $\phi_B^{-1}(0) = M_B$, $\phi_C^{-1}(0) = X$ (where $X \in \mathfrak{M}_C^0$). We shall show $\mu(M_A) = \nu(M_B) = X$. Indeed, since $u^2 - u \in M_D$, we see that for $a \in A$, $b \in B$, $c \in C$,

$$[(abcu)]^\wedge(M_A) = [(acu)]^\wedge(M_A)[(bu)]^\wedge(M_A)$$

$$= \phi_A(ac)\phi_B(b)$$

and

$$\phi_A(ac) = [(acu)]^\wedge(M_D) = [(au)]^\wedge(M_D)[(cu)]^\wedge(M_D)$$

$$= \phi_A(a)\phi_C(c).$$

Thus, if $\phi_A(a) \neq 0$, $\phi_A(ac)/\phi_A(a) = \phi_C(c)$. Similarly $\phi_B(bc) = \phi_B(b)\phi_C(c)$. In summary, for $z \in D$, and z represented by

$$\sum_{n=1}^{\infty} c_n(a_n \otimes b_n),$$

we have the formula

$$\hat{z}(M_D) = \sum_{n=1}^{\infty} \hat{c}_n(X)\hat{a}_n(M_A)\hat{b}_n(M_B).$$

Note that in the above $X \in \mathfrak{M}_C$, since otherwise $\hat{z}(M_D) = 0$ for all z, and this contradicts the fact that $M_D \in \mathfrak{M}_D$. Thus we see that an $M_D \in \mathfrak{M}_D$ engenders a pair $(M_A, M_B) \in \mathfrak{M}_A \times \mathfrak{M}_B$ such that $\rho(M_A, M_B) \in \Delta$. (Referring again to Example 5, we see that if C is contained in the radicals of both algebras A and B, then $\mathfrak{M}_D$ is empty.) Clearly if $(M_A, M_B) \in \rho^{-1}(\Delta)$ and if $\mu(M_A) = \nu(M_B) = M_C$ then the mapping taking $z \in D$ into

$$\sum_{n=1}^{\infty} \hat{c}_n(M_C)\hat{a}_n(M_A)\hat{b}_n(M_B)$$

(where z is represented by $\sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$) provides a nontrivial homomorphism $D \to C$. Let $\tau(M_D) = (M_A, M_B)$. We must now show that τ is a homeomorphism.

τ is 1-1: If $\tau(M_D) = \tau(M_D')$, then for any $z \in D$ represented by $c(a \otimes b)$ we have $\hat{z}(M_D) = \hat{z}(M_D')$, whence by linear extension, for any $z \in D$, $\hat{z}(M_D) = \hat{z}(M_D')$, and thus $M_D = M_D'$. Bicontinuity of τ is established as in [4] and in the corrected version in [5].

REMARK. $\overline{\rho^{-1}(\Delta)} \subset \rho^{-1}(\Delta) \cup \{(M_A, M_B) \mid \mu(M_A) = \nu(M_B) = \text{null map}\}$. Indeed, if $(M_A, M_B) \in \overline{\rho^{-1}(\Delta)}$ and $\mu(M_A) \neq \text{null map}$ or $\nu(M_B) \neq \text{null map}$, we may assume $\mu(M_A) \neq \nu(M_B)$ and $\mu(M_A) \neq \text{null map}$. Thus we may choose $c \in \nu(M_B) \setminus \mu(M_A)$ and so that $\hat{c}(\mu(M_A)) = 1$. Let $\hat{a}(M_A) = \hat{b}(M_B) = 1$ and choose

$$N(M_A, M_B) = \{(M_A', M_B') \mid \left| \hat{a}(M_A) - \hat{a}(M_A') \right| < \varepsilon, \ \left| \hat{b}(M_B) - \hat{b}(M_B') \right| < \varepsilon,$$

$$\left| [(ca)]\hat{\ }(M_A') - [(ca)]\hat{\ }(M_A') \right| < \varepsilon, \ \left| [(cb)]\hat{\ }(M_B) - [(cb)]\hat{\ }(M_B') \right| < \varepsilon \}.$$

Since $N(M_A, M_B) \cap \rho^{-1}(\Delta) \neq \varnothing$, there is in $N(M_A, M_B)$ a pair (M_A', M_B') such that $\mu(M_A') = \nu(M_B') = M_C' \neq \text{null map}$. However $[(ca)]\hat{\ }(M_A') = 1$, $([(cb)]\hat{\ }(M_B') = 0$,

$$\left| \hat{a}(M_A') \right| < 1 + \varepsilon, \ \left| \hat{b}(M_B') \right| > 1 - \varepsilon.$$

Hence, by direct calculation $(1 - \varepsilon)/(1 + \varepsilon) < \left| \hat{c}(M_C') \right| < \varepsilon/(1 - \varepsilon)$, an impossibility if $0 < \varepsilon < 1/3$. Thus $\overline{\rho^{-1}(\Delta)} \subset \rho^{-1}(\Delta) \cup \{(M_A, M_B) \mid \mu(M_A) = \nu(M_B) = \text{null map}\}$.

We have noted earlier, in connection with Example 5, that $\rho(\mathfrak{M}_A \times \mathfrak{M}_B)$ may

be $(\mathfrak{M}_C^0 \setminus \mathfrak{M}_C) \times (\mathfrak{M}_C^0 \setminus \mathfrak{M}_C)$, i.e., the null maps. Hence $\rho(\mathfrak{M}_A \times \mathfrak{M}_B)$ does not necessarily contain Δ. However we can prove

PROPOSTION 1. $\rho(\mathfrak{M}_A \times \mathfrak{M}_B) \supset \Delta$ *if and only if for every* $M_C \in \mathfrak{M}_C$, $I(M_C) = \{z \mid z \in D, z$ *represented by* $\sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$, $c_n \in M_C$, $n = 1, 2, \cdots\}$ *is contained in some* $M_D \in \mathfrak{M}_D$, *i.e., there is a pair* (M_A, M_B) *such that* $\mu(M_A) = \nu(M_B)$ $= M_C$ *if and only if* $I(M_C)$ *is contained in some* $M_D \in \mathfrak{M}_D$.

Proof. Assume $I(M_C)$ is contained in some $M_D \in \mathfrak{M}_D$: $I(M_C) \subset M_D$. Let $\tau(M_D) = (M_A, M_B)$, $\mu(M_A) = \nu(M_B) = M_C'$. We show $M_C = M_C'$. If $M_C \neq M_C'$, choose $c \in M_C \setminus M_C'$, $a \notin M_A$, $b \notin M_B$. Then w represented by $c(a \otimes b)$ satisfies

$$\hat{w}(M_D) = \hat{c}(M_C')\hat{a}(M_A)\hat{b}(M_B) \neq 0.$$

On the other hand $w \in M_D$ and a contradiction results.

If $\mu(M_A) = \nu(M_B) = M_C$, let $M_D = \tau^{-1}(M_A, M_B)$. Then if z, represented by

$$\sum_{n=1}^{\infty} c_n(a_n \otimes b_n),$$

is in $I(M_C)$,

$$\hat{z}(M_D) = \sum_{n=1}^{\infty} \hat{c}_n(M_C)\hat{a}_n(M_A)\hat{b}_n(M_B) = 0.$$

Hence $I(M_C) \subset M_D$.

If A, B, C are as in Example 5, then for $M_D \in \mathfrak{M}_D$, $\tau(M_D) = (M_A, M_B)$ where $\mu(M_A) = \nu(M_B) = M_C$ and M_C is the kernel of both homomorphisms $A \rightarrow A/M_A$, $B \rightarrow B/M_B$ when they are confined to C. In fact $[(ca)]^{\wedge}(M_A) = \hat{c}(M_A)\hat{a}(M_A)$. Thus $k(c, M_A) = \hat{c}(M_A)$ and we see $\hat{c}(M_C) = \hat{c}(M_A)$. Similarly, $\hat{c}(M_C) = \hat{c}(M_B)$.

The discussion of the semisimplicity of D when A, B, C are semisimple will not be given. The case where $C = C$ is to the writer's knowledge unresolved and *a fortiori* the general case remains an open problem. We shall see later how the semisimplicity of D enters in an essential fashion in the study of the area associated with Example 2.

3. Group algebras. We now turn to a more detailed discussion of Example 2. Direct integration computations show that A and B are C-bimodules in the sense discussed. We shall in this section write $\hat{G}$, $\hat{H}$, $\hat{K}$ for $\mathfrak{M}_A, \mathfrak{M}_B, \mathfrak{M}_C$ and α, β, γ for M_A, M_B, M_C [10]. One purpose of the development will be the discussion of the relationship of D to the group algebra $L_1(\mathfrak{G})$ of a group $\mathfrak{G}$ constructible from G, H, K. To this end we proceed as follows:

LEMMA 1.   *The maps* $\mu: \hat{G} \rightarrow \hat{K}$ *and* $\nu: \hat{H} \rightarrow \hat{K}$ *are the duals* θ_G^* *amd* θ_H^* *of the maps* θ_G *and* θ_H.

Proof.

$$[ca]^\wedge(\alpha) = \hat{c}(\mu(\alpha))\hat{a}(\alpha)$$

$$= \int_G \int_K a(\xi - \theta_G(\zeta))c(\zeta)\overline{(\xi,\alpha)} \, d\zeta d\xi$$

$$= \hat{a}(\alpha)\hat{c}(\theta_G^*(\alpha)).$$

Thus $\theta_G^*(\alpha) = \mu(\alpha)$ and similarly $\theta_H^*(\beta) = \nu(\beta)$.

It follows that

$$\tau(\mathfrak{M}_D) = \{(\alpha,\beta) \mid \theta_G^*(\alpha) = \theta_H^*(\beta)\}$$

$$= (\theta_G^* \times \theta_H^*)^{-1}(\text{diagonal } (\hat{K} \times \hat{K})),$$

a closed subgroup of $\hat{G} \times \hat{H}$. If indeed $D = L_1(\mathfrak{G})$ for some locally compact abelian group $\mathfrak{G}$ then $\mathfrak{M}_D$ and $\hat{\mathfrak{G}}$ are identifiable and $\mathfrak{M}_D = \hat{\mathfrak{G}}$. Since $\mathfrak{M}_D$ is already exhibited as a group we find

$$\hat{\mathfrak{M}}_D \cong (G \times H)/\mathfrak{M}_D^\perp$$

where $\mathfrak{M}_D^\perp = \{(\xi,\eta) \mid ((\alpha,\beta),(\xi,\eta)) = 1, \text{ all } (\alpha,\beta) \in \mathfrak{M}_D\}$ and the candidate for $\mathfrak{G}$ is $(G \times H)/\mathfrak{M}_D^\perp$.

LEMMA 2. *Let $\tilde{\theta}_H: K \to H$ denote the mapping $\zeta \to -\theta_H(\zeta)$. Then $\mathfrak{M}_D^\perp = (\theta_G \times \tilde{\theta}_H)(\text{diagonal } (K \times K)) \equiv Q$.*

Proof. Q is a closed subgroup of $G \times H$. For if for some net $(\theta_G(\zeta_\rho), \tilde{\theta}_H(\zeta_\rho)) \to (\xi,\eta) \in G \times H$, then from the fact that $\theta_G(K)$ and $\tilde{\theta}_H(K)$ are closed and the mappings $\theta_G: K \to \theta_G(K)$, $\tilde{\theta}_H: K \to \tilde{\theta}_H(K)$ are homomorphisms, hence open, we conclude $(\xi,\eta) \in \theta_G(K) \times \tilde{\theta}_H(K)$. Let $\theta_G(\zeta_1) = \xi$, $\tilde{\theta}_H(\zeta_2) = \eta$. Using the fact that $\theta_G: K \to \theta_G(K)$, $\tilde{\theta}_H: K \to \tilde{\theta}_H(K)$ are open, we can find a subnet $\zeta'_{\sigma_1} \to \zeta_1$, and a subsubnet $\zeta''_{\sigma_2} \to \zeta_2$ such that $\theta_G(\zeta'_{\sigma_1}) \to \xi$, $\tilde{\theta}_H(\zeta''_{\sigma_2}) \to \eta$. Since ζ''_{σ_1} is a subsubnet of ζ'_{σ_2}, and K is a T_2-space, we conclude $\zeta_1 = \zeta_2$ and $(\xi,\eta) \in Q$.

Now if $\xi = \theta_G(\zeta)$, $\eta = \tilde{\theta}_H(\zeta)$, then for $(\alpha,\beta) \in \mathfrak{M}_D$

$$(\theta_G(\zeta),\alpha)(\tilde{\theta}_H(\zeta),\beta) = (\zeta,\theta_G^*(\alpha))\overline{(\zeta,\theta_H^*(\beta))} = 1.$$

Hence $Q \subset \mathfrak{M}_D^\perp$. On the other hand, if $(\xi,\eta) \in \mathfrak{M}_D^\perp \setminus Q$, choose (α_0,β_0) such that $(\xi,\alpha_0)(\eta,\beta_0) \neq 1$ and $(\theta_G(\zeta),\alpha_0)(\tilde{\theta}_H(\zeta),\beta_0) = 1$, all $\zeta \in K$. Then for all α,β such that $\theta_G^*(\alpha) = \theta_H^*(\beta)$ (i.e., $(\alpha,\beta) \in \mathfrak{M}_D$)

$$(\xi,\alpha)(\eta,\beta) = 1.$$

But, by assumption $(\zeta,\theta_G^*(\alpha_0)) = (\zeta,\theta_H^*(\beta_0))$, all $\zeta \in K$. Hence $\theta_G^*(\alpha_0) = \theta_H^*(\beta_0)$ and thus $(\xi,\alpha_0)(\eta,\beta_0) = 1$, a contradiction. Thus $\mathfrak{M}_D^\perp = Q$. The mapping $\theta_G \times \tilde{\theta}_H$: diagonal $(K \times K) \to \mathfrak{M}_D^\perp$ is easily shown to be continuous and open. (Here and earlier we have exploited the standard proposition Let $\Phi: X \to Y$ be:

a surjection. Then Φ is open if and only if whenever $y_\rho = \Phi(x_\rho) \to \Phi(x) = y$ for some nets y_ρ, x_ρ, there is a net $x'_\sigma \to x$ and a subnet $y'_{\rho(\sigma)}$ of y_ρ such that $\Phi(x'_\sigma) = y_{\rho(\gamma)} \to y$.) Manifestly $\theta_G \times \tilde{\theta}_H$ is a homomorphism.

Presumably then $\mathfrak{G}$ should be $\widehat{\mathfrak{M}}_D$, i.e., if $\mathfrak{G} = (G \times H)/(\theta_G \times \tilde{\theta}_H)$(diagonal $(K \times K)$), then $D \cong L_1(\mathfrak{G})$. We shall see to what extent we can establish the truth of

$$D \cong L_1(\mathfrak{G}).$$

We shall assume in what follows that K is compact.

The first step in the process will be the construction of an epimorphism $T: D \to L_1(\mathfrak{G})$. It will then be shown that T is an isomorphism if and only if D is semisimple.

We begin by defining two mappings on the functions f of $F_C(A, B)$ with finite support:

$$T_1(c(a \otimes b))(\xi, \eta)$$

$$= \int_K \int_K a(\xi - \theta_G(\zeta_1) - \theta_G(\zeta_2))c(\zeta_1)b(\eta - \tilde{\theta}_H(\zeta_2)) \, d\zeta_1 d\zeta_2,$$

$$T_2(c(a \otimes b))(\xi, \eta)$$

$$= \int_K \int_K a(\xi - \theta_G(\zeta_1))c(\zeta_2)b(\eta - \theta_H(\zeta_2) - \tilde{\theta}_H(\zeta_1)) \, d\zeta_1 d\zeta_2.$$

By changing variables, $\zeta_2 \to \zeta_3 - \zeta_1$, we conclude $T_1 = T_2 \equiv \tilde{T}$. The usual integration procedures show $\tilde{T}(f) \in L_1(\mathfrak{G})$ and that $\tilde{T}$ is bounded. That is, although $\tilde{T}(f)$ is ostensibly a function of (ξ, η), actually $\tilde{T}(f)$ is constant on cosets of Q, $\tilde{T}(f) \in L_1(\mathfrak{G})$ and for some constant M, $\|\tilde{T}(f)\| \leq M\gamma_1(f)$. Thus $\tilde{T}$ may be extended to all $F_C(A, B)$ by continuity. Further manipulation of integrals shows that $\tilde{T}(I) = 0$ (since these manipulations are typified in some of the proofs below, they are here omitted). Hence $\tilde{T}$ defines a unique bounded mapping $T: D \to L_1(\mathfrak{G})$.

LEMMA 3. *T is multiplicative.*

Proof. Let $\mathfrak{g}$ represent the typical element of $\mathfrak{G}$: $\mathfrak{g} = (\xi, \eta)/Q$.

$$T(c_1 c_2(a_1 a_2 \otimes b_1 b_2))(\mathfrak{g})$$

$$= \int_G \int_H \int_K \int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2))a_2(\xi_1) c_1(\zeta_1 - \zeta_3) c_2(\zeta_3)$$

$$\cdot b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2))b_2(\eta_1) \, d\xi_1 d\eta_1 d\zeta_1 d\zeta_2 d\zeta_3.$$

This may be written after the change $\xi_1 \to \xi_1 - \theta_G(\zeta_3)$

$$\int_G \int_H \left[\int_K \int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2) + \theta_G(\zeta_3)) a_2(\xi_1 - \theta_G(\zeta_3)) \right.$$

$$\left. \cdot c_1(\zeta_1 - \zeta_3) c_2(\zeta_3) b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2)) b_2(\eta_1) \; d\zeta_1 d\zeta_2 d\zeta_3 \right] \; d\xi_1 d\eta_1.$$

and this in turn is

$$\int_{\mathfrak{G}} \left\{ \int_K \left[\int_K \int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2) + \theta_G(\zeta_3) - \theta_G(\zeta_4)) \right. \right.$$

$$\left. \cdot a_2(\xi_1 - \theta_G(\zeta_3) - \theta_G(\zeta_4)) c_1(\zeta_1 - \zeta_3) c_2(\zeta_3) b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2) - \tilde{\theta}_H(\zeta_4)) \right.$$

$$\left. \left. \cdot b_2(\eta_1 - \tilde{\theta}_H(\zeta_4)) \; d\zeta_1 d\zeta_2 d\zeta_3 \right] d\zeta_4 \right\} d\mathfrak{g}.$$

We have used the formula

$$\int_Q \Phi(\omega) \; d\omega \; = \; \int_{\text{diagonal}(K \times K)} \Phi((\theta_G \times \tilde{\theta}_H)(\zeta, \zeta)) d(\zeta, \zeta) = \int_K \Phi((\theta_G \times \tilde{\theta}_H)(\zeta, \zeta)) \; d\zeta,$$

valid (if the Haar measures are adjusted) for any Φ continuous and with compact support. The formula follows from the simple observation: If $\theta : X \to Y$ is an epimorphism of compact groups, then

$$\int_X \Phi(\theta(x)) \; dx$$

is an invariant functional on $L_1(Y)$. Thus for some Haar measure on Y

$$\int_X \Phi(\theta(x)) \; dx \; = \; \int_Y \Phi(y) \; dy.$$

We assume the adjustment has been made. We find then

$$T(c_1(a_1 \otimes b_1)) T(c_2(a_2 \otimes b_2))(\xi, \eta)$$

$$= \int_{\mathfrak{G}} \left\{ \left[\int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2)) c_1(\zeta_1) b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2)) d\zeta_1 d\zeta_2 \right] \right.$$

$$\left. \cdot \left[\int_K \int_K a_2(\xi_1 - \theta_G(\zeta_3) - \theta_G(\zeta_4)) c_2(\zeta_3) b_2(\eta_1 - \tilde{\theta}_H(\zeta_4)) d\zeta_3 d\zeta_4 \right] \right\} d\mathfrak{g}.$$

Changing variables ($\zeta_1 \to \zeta_1 - \zeta_3$) in the first [ ] above we obtain

$$\int_{\mathfrak{G}} \left\{ \left[\int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2) + \theta_G(\zeta_3)) c_1(\zeta_1 - \zeta_3) \right. \right.$$

$$\left. \cdot b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2)) \; d\zeta_1 d\zeta_2 \right]$$

$$\left. \cdot \left[\int_K \int_K a_2(\xi_1 - \theta_G(\zeta_2) - \theta_G(\zeta_4)) c_2(\zeta_3) b_2(\eta_1 - \tilde{\theta}_H(\zeta_4)) d\zeta_3 d\zeta_4 \right] \right\} d\mathfrak{g}.$$

We may replace (ξ, η) by $(\xi, \eta) - (\theta_G(\zeta_4), \tilde{\theta}_H(\zeta_4))$ in the first $[\quad]$ above and obtain

$$\int_{\mathfrak{G}} \left\{ \left[\int_K \int_K a_1(\xi - \xi_1 - \theta_G(\zeta_1) - \theta_G(\zeta_2) + \theta_G(\zeta_3) - \theta_G(\zeta_4))c_1(\zeta_1 - \zeta_3) \right.\right.$$
$$\left. \cdot\, b_1(\eta - \eta_1 - \tilde{\theta}_H(\zeta_2) - \tilde{\theta}_H(\zeta_4))\, d\zeta_1 d\zeta_2 \right]$$
$$\left. \cdot \left[\int_K \int_K a_2(\xi_1 - \theta_G(\zeta_3) - \theta_G(\zeta_4))c_2(\zeta_3)b_2(\eta_1 - \tilde{\theta}_H(\zeta_4))\, d\zeta_3 d\zeta_4 \right] \right\}\; dg.$$

Comparison of the formulas for $T(c_1 c_2(a_1 a_2 \otimes b_1 b_2))$ and $T(c_1(a_1 \otimes b_1))$ $\cdot T(c_2(a_2 \otimes b_2))$ shows they are the same and we conclude that T is multiplicative.

LEMMA 4. *Let $\tilde{A}$, $\tilde{C}$ be commutative Banach algebras, $\tilde{A}$ a $\tilde{C}$-bimodule. Let $\{\tilde{c}\}$ be an approximate identity for $\tilde{C}$. Then $\{\tilde{c}\}$ is also an approximate identity for $\tilde{A}$ if and only if each $\tilde{a}_1 \in \tilde{A}$ is of the form $\tilde{c}_1 \tilde{a}_2$, $\tilde{c}_1 \in \tilde{C}, \tilde{a}_2 \in \tilde{A}$.*

Proof. Paraphrasing Cohen's argument [3], we find that if $\{\tilde{c}\}$ is an approximate identity for $\tilde{A}$, then each $\tilde{a}_1 \in \tilde{A}$ is of the form indicated. Conversely if $\tilde{a}_1 \in \tilde{A}$, let $\tilde{a}_1 = \tilde{c}_1 \tilde{a}_2$. Then $\tilde{c}\tilde{a}_1 - \tilde{a}_1 = \tilde{c}\tilde{c}_1 \tilde{a}_2 - \tilde{c}_1 \tilde{a}_2 = (\tilde{c}\tilde{c}_1 - \tilde{c}_1)\tilde{a}_2 \to 0$ on the net $\{\tilde{c}\}$.

LEMMA 5. *Let K be compact and let $\{c\}$, where all c are continuous and nonnegative, be an approximate identity in $C = L_1(K)$. Then for $a \in A = L_1(G)$,*

$$a_c(\xi) = \int_K a(\xi - \theta_G(\zeta))c(\zeta)\, d\zeta \to a$$

on the net $\{c\}$.

Proof. We may assume that a is continuous, nonnegative and that a has compact support. Hence there is a fixed compact $\Lambda \subset G$ such that a and all a_c are 0 off Λ. Then we write

$$\int_G \left| a(\xi) - a_c(\xi) \right|\, d\xi$$
$$= M_1 \int_{G/\theta_G(K)} \left(\int_K \left| a(\xi + \theta_G(\zeta)) - a_c(\xi + \theta_G(\zeta)) \right|\, d\zeta \right) d\lambda,$$

where $\lambda = \xi/\theta_G(K)$ and where M_1 is some constant. (An adjustment of Haar measure would make $M_1 = 1$. This would have to be reconciled with the earlier adjustment used in the proof of Lemma 3. For our purposes, the above suffices.) Our assumptions imply that the functions

$$\int_K a(\xi + \theta_G(\zeta))\, d\zeta, \quad \int_K a_c(\xi + \theta_G(\zeta))\, d\zeta,$$

in $L_1(G/\theta_G(K))$ are continuous and that they all vanish off the compact set $\Lambda/\theta_G(K)$.

For $\varepsilon > 0$, and each $\xi + \theta_G(K) \in \Lambda/\theta_G(K)$, choose $\delta_1 > 0$ and c_ξ so that

(*) $$\int_K \big| a(\xi + \theta_G(\zeta)) - a_c(\xi + \theta_G(\zeta)) \big| \, d\zeta < \delta_1$$

for c_ξ and all c beyond c_ξ. Let $\delta_2 > 0$ and let V be a neighborhood of 0 in $G/\theta_G(K)$ such that for $\xi_1 + \theta_G(K) \in \xi + \theta_G(K) + V$, $\int_K \big| a(\xi + \theta_G(\zeta)) - a(\xi_1 + \theta_G(\zeta)) \big| \, d\zeta < \delta_2$. Then, for such ξ, ξ_1,

$$\left| \int_K (a_c(\xi + \theta_G(\zeta)) - a_c(\xi_1 + \theta_G(\zeta)) \, d\zeta \right|$$

$$= \left| \int_K \int_K (a(\xi + \theta_G(\zeta) - \theta_G(\zeta_1)) - a(\xi_1 + \theta_G(\zeta) - \theta_G(\zeta_1))) \, c(\zeta_1) \, d\zeta \, d\zeta_1 \right|$$

$$\leq \int_K |c(\zeta_1)| \left(\int_K |a(\xi + \theta_G(\zeta)) - a(\xi_1 + \theta_G(\zeta))| \, d\zeta \right) d\zeta_1$$

$$< \delta_2.$$

In other words, the functions $\int_K a(\xi + \theta_G(\zeta)) \, d\zeta$, $\int_K a_c(\xi + \theta_G(\zeta)) \, d\zeta$ form an equicontinuous family. Hence for each $\xi + \theta_G(K) \in \Lambda/\theta_G(K)$ there is a neighborhood $V(\xi + \theta_G(K))$ where the inequality (*) persists for all c beyond c_ξ. Let $V(\xi_i + \theta_G(K))$, $i = 1, 2, \cdots, N$ be a finite subcovering of $\Lambda/\theta_G(K)$. Then for all c beyond the "last" c_{ξ_i} and for any $\xi + \theta_G(K) \in \Lambda/\theta_G(K)$, (*) is true. Hence if

$$\delta_1 M_1 \int_{\Lambda/\theta_G(K)} d\lambda < \varepsilon,$$

we see $\int_G \big| a(\xi) - a_c(\xi) \big| \, d\xi < \varepsilon$ for all c beyond the "last" c_{ξ_i}, i.e.,

$$\lim_{\{c\}} \int_G \big| a(\xi) - a_c(\xi) \big| \, d\xi = 0.$$

Furthermore, since $\| c \| \equiv 1$, we find $\| a_c \| \leq \| a \|$.

Combining Lemmas 4 and 5 we see that there is an approximate identity $\{c\}$ for C that is also an approximate identity for A, and *mutatis mutandis* for B.

LEMMA 6.   *T is surjective.*

Proof.   Let $\phi \in L_1(\mathfrak{G})$. If $R: L_1(G \times H) \to L_1(\mathfrak{G})$ is the canonical epimorphism

$$R(\psi) = \int_Q \psi((\xi, \eta) + q) \, dq,$$

let $R(\Phi) = \phi$. Via [7, p. 59–60], [9] we can find a $\tilde{z} \in A \otimes_\gamma B$, $\tilde{z}$ represented by

$$\sum_{n=1}^{\infty} (\tilde{a}_n \otimes b_n),$$

such that in the isomorphism $A \otimes_\gamma B \leftrightarrow L_1(G \times H)$, $\tilde{z} \leftrightarrow \Phi$. Exploiting Lemmas

4 and 5 we find elements $c_n \in C$ and $a_n \in A$ such that $\tilde{a}_n = a_n c_n$. It is now clear that if z in D is represented by

$$\sum_{n=1}^{\infty} c_n(a_n \otimes b_n),$$

then $T(z) = \phi$.

THEOREM 1.   *T is an isomorphism if and only if D is semisimple.*

Since $\mathfrak{G}$ is a locally compact abelian group, $L_1(\mathfrak{G})$ is semisimple and thus if T is an isomorphism, D is semisimple.

On the other hand, let D be semisimple. We shall show that for any $z \in T^{-1}(0)$, $\hat{z}(M_D) \equiv 0$, whence $z = 0$, T is 1-1 and, from the earlier results, an isomorphism. To this end let

$$\sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$$

represent $z \in T^{-1}(0)$. For any M_D let $\tau(M_D) = (\alpha, \beta)$ where $\theta_G^*(\alpha) = \theta_H^*(\beta) = \gamma \in \hat{K}$. Then

$$\hat{z}(M_D) = \sum_{n=1}^{\infty} \hat{c}_n(\gamma)\hat{a}_n(\alpha)\hat{b}_n(\beta)$$

$$= \sum_{n=1}^{\infty} \int_{\mathfrak{G}}\int_{H} \left(\int_{K} a_n(\xi - \theta_G(\zeta_1))c_n(\zeta_1)b_n(\eta)\,d\zeta_1 \right)\overline{(\xi,\alpha)(\eta,\beta)}\ d\xi d\eta$$

$$= \sum_{n=1}^{\infty} \int_{\mathfrak{G}} \left\{ \int\int_{K}\left[\int_{K} a_n(\xi + \theta_G(\zeta_2) - \theta_G(\zeta_1))c_n(\zeta_1)b_n(\eta + \tilde{\theta}_H(\zeta_2))\ d\zeta_1 \right] \right.$$
$$\left. \cdot \overline{(\xi + \theta_G(\zeta_2),\alpha)}\ \overline{(\eta + \tilde{\theta}_H(\zeta_2),\beta)}\ d\zeta_2 \right\}\ dg$$

where g denotes the coset of (ξ, η) in $\mathfrak{G}$,

$$(\xi + \theta_G(\zeta_2),\alpha)(\eta + \tilde{\theta}_H(\zeta_2),\beta) = (g, \{\alpha,\beta\})$$

where $\{\alpha, \beta\}$ is the element of $\hat{\mathfrak{G}}$ corresponding to the pair $(\alpha, \beta) \in Q^{\perp}$. Setting $T(c_n(a_n \otimes b_n)) = \psi_n(g)$ we find the above sum of integrals reduces to

$$\sum_{n=1}^{\infty} \int_{\mathfrak{G}} \psi_n(g)(g, \{\alpha,\beta\})\ dg = [(Tz)]^{\wedge}\{\alpha,\beta\} \equiv 0.$$

Hence $z = 0$ and T is an isomorphism.

Note that when $C = C$, D is semisimple, $\mathfrak{G} = G \times H$ and a known result [5], [9] obtains: $L_1(G) \otimes_\gamma L_1(H) = L_1(G \times H)$.

In [6] the relationship between Grothendieck's "condition of approximation" and its equivalents [7] on the one hand and the semisimplicity of $A \otimes_\gamma B$, when A and B are semisimple, on the other hand were examined.

Pursuit of the same line here does not appear promising. The method that succeeds in the case of $A \otimes_\gamma B$ would require, in the context of $L_1(G) \otimes_{L_1(K)} L_1(H)$, the presence of many finite-dimensional $L_1(K)$-modules in $L_1(G)$. However, if $G = \mathbf{R}$, $K = \mathbf{Z}$ and if θ_G is the inclusion map, then $L_1(G)$ has no finite-dimensional $L_1(K)$-modules other than $\{0\}$. This fact can be verified quite easily as follows:

Let N be a nontrivial finite-dimensional $L_1(\mathbf{Z})$-module in $L_1(\mathbf{R})$ and let $\{a_i\}_{i=1}^n$ be a basis for N. For some integer r, real numbers t_1, t_2, t_3, t_4, and some c in $L_1(\mathbf{Z})$ the following hold:

(a) $r < t_1 < t_2 < t_3 < t_4 < r + 1$;

(b) $\hat{c}(\tilde{t}) = 0$ if $0 < \tilde{t}_1 < \tilde{t} < \tilde{t}_2 < 1$, $\hat{c}(\tilde{t}) = 1$ if $0 < \tilde{t}_3 < \tilde{t} < \tilde{t}_4 < 1$, where $\tilde{t} = t \pmod{1}$;

(c) $\{\hat{a}_i(\tilde{t})\}$ are linearly independent on $(\tilde{t}_1, \tilde{t}_2)$ and on $(\tilde{t}_3, \tilde{t}_4)$.

If $ca_i = \sum_{j=1}^n \lambda_{ij} a_j$ we rapidly conclude the contradiction: $\lambda_{ij} = \delta_{ij} = 0$ for all $i, j = 1, 2, \cdots, n$.

4. **Tensor products of groups.**   In the setting most briefly described by the symbols $\{G, H, K, \theta_G, \theta_H\}$ we may interpret K as operating on each of G and H according to the following format: Let $G = \{\xi\}$, $H = \{\eta\}$, $K = \{\zeta\}$. Define the action of $\zeta \in K$ on $\xi \in G$ by $\zeta(\xi) = \xi + \theta_G(\zeta)$. Similarly $\zeta(\eta) = \eta + \theta_H(\zeta)$. Thus each $\zeta \in K$ induces 1-1 mappings of G and H upon themselves.

It is simple to give abstract characterizations of the actions of these mappings. Then, in $G \times H$ we may introduce a relation S by the rule: $(\xi_1, \eta_1) S (\xi_2, \eta_2)$ if and only if there is a $\zeta \in K$ such that $\zeta(\xi_1) = \xi_2$ and $\zeta(\eta_2) = \eta_1$. The relation S is easily shown to be an equivalence relation because K is abelian. The S-equivalence class containing $(0_G, 0_H)$, the identity of $G \times H$, is then readily seen to be a subgroup (normal since G and H are abelian) which we may readily identify with what we have already denoted by Q. In terms of the mappings that K induces on G and H, we may then recover θ_G and θ_H by the formulas $\theta_G(\zeta) = \zeta(0_G)$, $\theta_H(\zeta) = \zeta(0_H)$.

In the light of the above we choose to interpret $G \times H / Q$ as the tensor product $G \otimes_K H$ relative to the mappings θ_G and θ_H or (equivalently) relative to the defined actions of K on G and H.

If we show then that $L_1(G) \otimes_{L_1(K)} L_1(H)$ and $L_1(\mathfrak{G})$ are isomorphic we shall simultaneously show that the tensor product of group algebras is the group algebra of the tensor product.

The theorem we shall prove is

THEOREM 1.   *If G and H are locally compact abelian groups and if K is a compact abelian group, then*

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(G \otimes_K H)$$

where the tensor products in question are taken with respect to continuous homomorphisms θ_G and θ_H of K into G and H.

This result was proved in 1962. During an unavoidable delay in the publication process, Boaz Natzitz had the opportunity to work with the author and he extended the theorem to the case where K is permitted to be locally compact (and abelian). His efforts led to a doctoral dissertation at the University of Minnesota, and his paper (incorporating new methods) will be published at a later date.

We begin the proof of the theorem with

LEMMA 1. *Let G be a locally compact abelian group and let $a \in L_1(G)$. Then for $\varepsilon > 0$ there is a $u \in L_1(G)$ such that $\|u\| \leq 2$, u is continuous, u has compact support and such that $\|a - au\| < \varepsilon$.*

Proof. Let $v \in L_1(G)$ be such that $\|a - av\| < \varepsilon/2$, $\|v\| = 1$. Choose u continuous in $L_1(G)$ and so that $\|u\| \leq 2$, $\|v - u\| < \varepsilon/2\|a\|$ (to avoid the trivial case we assume $a \neq 0$) and u has compact support [10]. Then $\|a - au\| \leq \|a - av\| + \|av - au\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Now let $\tilde{c}_\gamma = (\zeta, \gamma)$ where $\zeta \in K$, $\gamma \in \hat{K}$. Then $\tilde{c}_\gamma \in L_1(K)$. If $a \in L_1(G)$ and if $\hat{a}$ has compact support then

$$\tilde{c}_\gamma a = \int_K a(\xi - \theta_G(\zeta))(\zeta, \gamma) \, d\zeta$$

$$= \int_K \left(\int_{\hat{G}} \hat{a}(\alpha)(\xi - \theta_G(\zeta), \alpha) \, d\alpha \right)(\zeta, \gamma) \, d\zeta.$$

Hence, the fundamental relations among group characters of compact groups show

$$\tilde{c}_\gamma a = \int_{\hat{G}} \hat{a}(\alpha)(\xi, \alpha) \left(\int_K (\zeta, \gamma - \theta_G^*(\alpha)) \, d\zeta \right) d\alpha$$

$$= \int_{\theta_G^{*-1}(\gamma)} \hat{a}(\alpha)(\xi, \alpha) \, d\alpha.$$

Let $z'' \in D$ be represented by $\sum_{n=1}^{\infty} c_n''(a_n'' \otimes b_n'')$ and be such that $\hat{z}''(M_D) \equiv 0$. By the use of approximate identities (cf. Lemma 1) we can for $\varepsilon > 0$ produce a z' represented by $\sum_{n=1}^{\infty} c_n(a_n' \otimes b_n')$ such that $\gamma(z' - z'') < \varepsilon/2$, $\hat{z}'(M_D) \equiv 0$, and such that all $\hat{a}_n', \hat{b}_n', \hat{c}_n$ have compact support. Actually, and we shall need this in the course of our argument, there are fixed compact sets Γ, H, K in $\hat{G}, \hat{H}, \hat{K}$ such that for all n, support $(\hat{a}_n') \subset \Gamma$, support $(\hat{b}_n') \subset H$, support $(\hat{c}_n) \subset K$. Finally, by the use of two functions u, v, nonnegative and with compact supports, C_u, D_v we may write $a_n = ua_n'$, $b_n = vb_n'$ and conclude that $\sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$ represents an element z such that $\gamma(z' - z) < \varepsilon/2^2$, $\hat{z}(M_D) \equiv 0$. The exploitation of (3'),(4'), §1, permits us to assume $\sum_{n=1}^{\infty} \|a_n'\|$, $\sum_{n=1}^{\infty} \|b_n'\| < \infty$.

Since K is compact, $\hat{K}$ is discrete, K is finite and each $c_n = \sum_{\gamma \in K} \hat{c}_n(\gamma)(\zeta, \gamma)$. We conclude that

$$\sum_{n=1}^{\infty} \left(\sum_{\gamma \in K} \hat{c}_n(\gamma) \hat{c}_\gamma \right)$$

$$\cdot \left(\int_G \int_{\hat{G}} \hat{a}'_n(\alpha)(\xi_1, \alpha) u(\xi - \xi_1) \, d\xi_1 \, d\alpha \otimes \int_H \int_{\hat{H}} \hat{b}'_n(\beta)(\eta_1, \beta) v(\eta - \eta_1) \, d\eta_1 \, d\beta \right)$$

represents z.

Next we write for each $\gamma \in K$,

$$\int_{\hat{G}} = \int_{\theta_G^{*-1}(\gamma)} + \searrow \int_{\hat{G} \setminus \theta_G^{*-1}(\gamma)},$$

$$\int_{\hat{H}} = \int_{\theta_H^{*-1}(\gamma)} + \searrow \int_{\hat{H} \setminus \theta_H^{*-1}(\gamma)}.$$

From Lemma 1, §1 we see that if $cc'(a \otimes b)$ represents some w and $c'a = 0$, then $w = 0$. In the present circumstances, if $c(a \otimes b)$ represents some w and $ca = 0$, then $w = 0$. For if c' is properly chosen so that $\| c'c - c \|$ is small, then $cc'(a \otimes b)$ represents a w' near w and on the other hand $w' = 0$.

From the observations

$$\tilde{c}_\gamma a = \int_{\theta_G^{*-1}(\gamma)} \hat{a}(\alpha)(\xi, \alpha) \, d\alpha$$

and $\tilde{c}_\gamma * \tilde{c}_\gamma = \tilde{c}_\gamma$ we conclude that z is represented by

$$\sum_{n=1}^{\infty} \left(\sum_{\gamma \in K} \tilde{c}_n(\gamma) \tilde{c}_\gamma \right)$$

$$\cdot \left(\int_G \int_{\theta_G^{*-1}(\gamma)} \hat{a}'_n(\alpha)(\xi_1, \alpha) u(\xi - \xi_1) \, d\xi_1 d\alpha \otimes \int_H \int_{\theta_H^{*-1}(\gamma)} \hat{b}'_n(\beta)(\eta_1, \beta) v(\eta - \eta_1) \, d\eta_1 d\beta \right).$$

If z_N is represented by the first N terms of the above series, then $\gamma(z_N - z) \to 0$ as $N \to \infty$. We shall show that for each N, there are finitely many precompact disjoint measurable sets C_i, D_j in G, H, elements $\tilde{\xi}_i \in C_i$, $\tilde{\eta}_j \in D_j$ for which $\sum_i m_G(C_i)$, $\sum_j m_H(D_j) < T_0 < \infty$, where T_0 is N-free, and for which

$$\sum_{\gamma \in K} \left(\sum_{i,j} \left(\int_{\theta_G^{*-1}(\gamma)} \int_{\theta_H^{*-1}(\gamma)} \sum_{n=1}^{N} \hat{c}_n(\gamma) \hat{a}'_n(\alpha) \hat{b}'_n(\beta)(\tilde{\xi}_i, \alpha)(\tilde{\eta}_j, \beta) \, d\alpha d\beta \right) \right.$$

$$\left. \cdot (m_G(C_i) u(\xi - \tilde{\xi}_i) \otimes m_H(D_j) v(\eta - \tilde{\eta}_j)) \right)$$

represents a $\tilde{z}_N$ such that $\gamma(\tilde{z}_N - z_N) < \varepsilon/2^4$, for all N. Assuming this we finish the proof as follows: First we see $\sum_{n=1}^{\infty} \hat{c}_n(\gamma) \hat{a}'_n(\alpha) \hat{b}'_n(\beta) = 0$ in the ranges of integration, and $S = \sum_{n=1}^{\infty} \| c_n \| \| a'_n \| \| b'_n \| < \infty$. Thus for some N_0 depending only on S and on the measure of $H \times K$

$$(**) \qquad \left\| \left(\int_{\theta_G^{*-1}(\gamma)} \int_{\theta_H^{*-1}(\gamma)} \sum_{n=1}^{N} \hat{c}_n(\gamma) \hat{a}'_n(\alpha) \hat{b}'_n(\beta)(\tilde{\xi}_i, \alpha)(\tilde{\eta}_j, \beta) \, d\alpha d\beta \right) \tilde{c}_\gamma \right\| < \delta$$

where $\delta \cdot m_K(K) \cdot T_0^2 < \varepsilon/2^5$, if $N \geq N_0$. If $r_i = m_G(C_i)$, $s_j = m_K(D_j)$, then

$$\gamma(\tilde{z}_N) \; < \; \delta \cdot m_K(K) \left(\sum_{i,j} r_i \| u(\xi - \xi_i) \| \, s_j \| v(\eta - \tilde{\eta}_j) \| \right)$$

$$= \; \delta \cdot m_K(K) \sum_{i,j} r_i s_j \; = \; \delta \cdot m_K(K) \left(\sum_i r_i \right) \left(\sum_j s_j \right)$$

$$< \; \delta \cdot m_K(K) \, T_0^2 < \varepsilon/2^5 \, .$$

Hence $\gamma(\tilde{z}_N) < \varepsilon/2^5$.

In summary, we choose N_0 so that (**) holds for $N \geq N_0$, then $N > N_0$ so that $\gamma(z_N - z) < \varepsilon/2^3$, then C_i, D_j, ξ_i, η_j so that $\gamma(\tilde{z}_N - z_N) < \varepsilon/2^4$. As above, $\gamma(\tilde{z}_N) < \varepsilon/2^5$ and

$$\gamma(z'') \leqq \gamma(z'' - z') + \gamma(z' - z) + \gamma(z - z_N) + \gamma(z_N - \tilde{z}_N) + \gamma(\tilde{z}_N) < \varepsilon \, .$$

Since $\varepsilon > 0$ is arbitrary, we see $z'' = 0$.

Thus we must prove the statement concerning the $r_i, s_j, \cdots$.

For brevity we set $\tilde{c}_j a_n' = d_{n\gamma}$, $\tilde{c}_\gamma b_n' = e_{n\gamma}$. Then z_N is represented by $\sum_{n=1}^N (\sum_{\gamma \in K} \hat{c}_n(\gamma) \tilde{z}_j)(d_{n\gamma} u \otimes e_{n\gamma} v)$. For $\rho > 0$ there are compact sets C_d, D_e such that

$$\int_{G \setminus C_d} \sum_{n=1}^{\infty} \sum_{\gamma \in K} | d_{n\gamma}(\xi) | \; d\xi, \quad \int_{H \setminus D_e} \sum_{n=1}^{\infty} \sum_{\gamma \in K} | e_{n\gamma}(\eta) | \; d\eta < \rho/2 \, ,$$

since $\sum_{n=1}^{\infty} \| a_n' \| \sum_{n=1}^{\infty} \| b_n' \| < \infty$. Hence there are compact sets $\bar{C}, \bar{D}$, *independent of N* such that w_N represented by

$$\sum_{n=1}^N \left(\sum_{\gamma \in K} \hat{c}_n(\gamma) \, \tilde{c}_\gamma \right) \left(\int_{\bar{C}} d_{n\gamma}(\xi_1) u(\xi - \xi_1) \; d\xi_1 \otimes \int_{\bar{D}} e_{n\gamma}(\eta_1) v(\eta - \eta_1) \, d\eta_1 \right)$$

satisfies $\gamma(z_N - w_N) < \rho/2^2$. The functions $d_{n\gamma}, e_{n\gamma}$ are uniformly continuous and thus for $\sigma > 0$ there are neighborhoods $X(0)$, $Y(0)$ in G, H, such that

$$| d_{n\gamma}(\xi_1) - d_{n\gamma}(\xi_2) |, | e_{n\gamma}(\eta_1) - e_{n\gamma}(\eta_2) |, \| u_{\xi_1} - u_{\xi_2} \|, \| v_{\eta_1} - v_{\eta_2} \| < \sigma$$

if $\xi_1 - \xi_2 \in X(0), \eta_1 - \eta_2 \in Y(0)$.

If we cover $\bar{C}$ and $\bar{D}$ with finitely many translates of $X(0)$ and $Y(0)$ and then partition $\bar{C}$ and $\bar{D}$ into finitely many measurable precompact sets C_i, D_j each of which is contained in one of the chosen translates of $X(0)$, $Y(0)$, and if $\tilde{\xi}_i \in C_i$, $\tilde{\eta}_j \in D_j$, we can estimate

$$\left\| \int_{\bar{C}} d_{n\gamma}(\xi_1) u(\xi - \xi_1) \; d\xi_1 - \sum_i d_{n\gamma}(\tilde{\xi}_i) u(\xi - \tilde{\xi}_i) m_G(C_i) \right\|,$$

$$\left\| \int_{\bar{D}} e_{n\gamma}(\eta_1) v(\eta - \eta_1) d\eta_1 - \sum_j e_{n\gamma}(\tilde{\eta}_j) v(\eta - \tilde{\eta}_j) m_H(D_j) \right\|.$$

Indeed (we estimate only one of the above, the other is treated similarly),

$$\left\| \int_{\bar{C}_i} - \sum_i \right\| = \left\| \sum_i \int_{C_i} - \sum_i \right\|$$

$$\leqq \left\| \sum_i \int_{C_i} d_{n\gamma}(\xi_1)(u(\xi - \xi_1) - u(\xi - \tilde{\xi}_i)) \, d\xi_1 \right\| + \sigma m_G(\bar{C}_i) \left\| u \right\|$$

$$\leqq \sum_i \int_{C_i} |d_{n\gamma}(\xi_1)| \left(\int_G |u(\xi - \xi_1) - u(\xi - \tilde{\xi}_i)| \, d\xi \right) d\xi_1 + \sigma m_G(\bar{C})$$

$$\leqq \sigma \left(\int |d_{n\gamma}(\xi_1)| \, d\xi_1 + m_G(\bar{C}) \right).$$

It is now clear that for $\sigma > 0$ small enough, $\tilde{w}_N$ represented by

$$\sum_{n=1}^{N} \left(\sum_{\gamma \in K} \hat{c}_n(\gamma)\tilde{c}_\gamma \right) \left(\sum_i d_{n\gamma}(\tilde{\xi}_i)u(\xi - \tilde{\xi}_i)m_G(C_i) \otimes \sum_j e_{n\gamma}(\tilde{\eta}_j)v(\eta - \tilde{\eta}_j)m_H(D_j) \right)$$

satisfies $\gamma(\tilde{w}_N - w_N) < \rho/2^2$. It is clear that

$$\sum_{\gamma \in K} \sum_{i,j} \left(\int_{\theta_G^{*-1}(\gamma)} \int_{\theta_H^{*-1}(\gamma)} \sum_{n=1}^{N} \hat{c}_n(\delta)\hat{a}'_n(\alpha)\hat{b}'_n(\beta)(\tilde{\xi}_i, \alpha)(\tilde{\eta}_j, \beta) \, d\alpha d\beta \, \tilde{c}_\gamma \right.$$

$$\left. \cdot (m_G(C_i)u(\xi - \tilde{\xi}_i) \otimes m_H(D_j)v(\eta - \tilde{\eta}_j)) \right).$$

represents $\tilde{w}_N$, and that $\sum_i m_G(C_i)$, $\sum_j m_H(D_j) \leqq \max(m_G(\bar{C}), m_H(\bar{D})) = T_0 < \infty$. Hence for small enough ρ, $\gamma(\tilde{z}_N - z_N) < \rho/2 + \rho/2^2 < \varepsilon/2^4$. The proof is over.

In the theory of group extensions [2] two fixed groups K, S are considered and the exact sequences

$$0 \to K \xrightarrow{\theta_G} G \xrightarrow{\phi_G} S \to 0$$

are gathered into equivalence classes. Two such sequences (one with G, one with H) are considered equivalent if and only if there is an isomorphism $\psi_{GH}: G \to H$ such that

$$0 \to K \underset{\theta_H}{\overset{\theta_G}{\rightarrow}} \begin{matrix} G \\ \downarrow \psi_{GH} \\ H \end{matrix} \underset{\phi_H}{\overset{\phi_G}{\rightarrow}} S \to 0$$

is a commutative diagram. If G and H are class representatives (the diagram may or may not be commutative), the group

$$J = [(\phi_G \times \phi_H)^{-1}(\text{diagonal}(S \times S))]/[(\theta_G \times \tilde{\theta}_H)(\text{diagonal}(K \times K))]$$

is again associated with an exact sequence

$$0 \rightarrow K \xrightarrow{\theta_J} J \xrightarrow{\phi_J} S \rightarrow 0$$

representative of the Baer product [1], [2] of the originally given classes.

In our discussion

$$\mathfrak{G} = G \times H / [(\theta_G \times \tilde{\theta}_H)(\text{diagonal}(K \times K))]$$

and

$$\hat{\mathfrak{G}} = [(\theta_G^* \times \theta_H^*)^{-1} \, (\text{diagonal}(\hat{K} \times \hat{K}))]^{\perp}.$$

In particular $\mathfrak{G}$ is a supergroup of a representative of the Baer product when $G/\theta_G(K)$ and $H/\theta_H(K)$ are isomorphic. It is a simple matter to verify that $\mathfrak{G}$ is a representative of the Baer product if and only if θ_G and θ_H are isomorphisms. In this case we obtain $\mathfrak{G} = G$, $L_1(G) \otimes_{L_1(G)} L_1(G) = L_1(G)$ a result derivable by direct examination of the data.

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